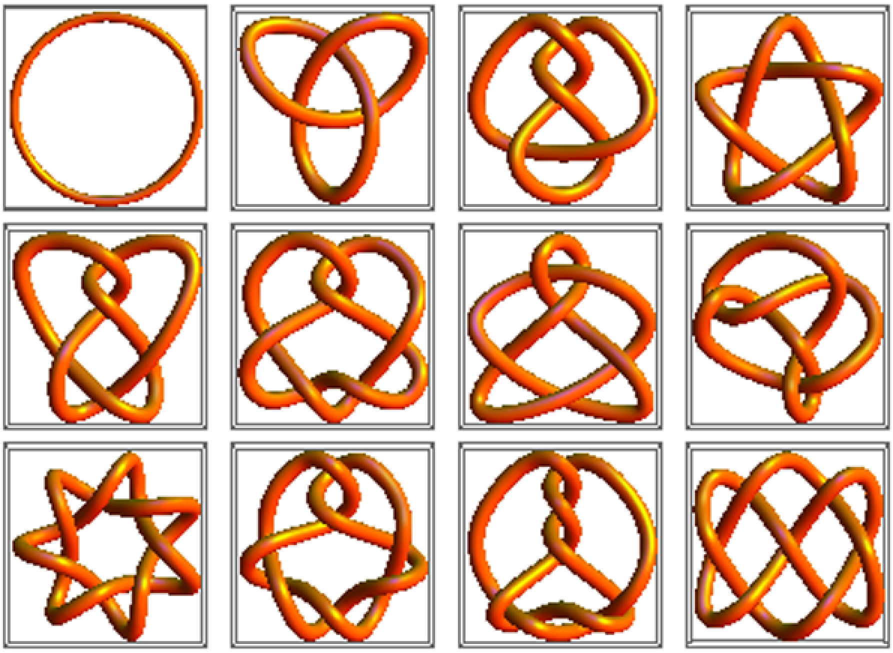
Qualitative Shape

Ellis D. Cooper, March 24, 2015

Mathematicians sometimes focus on how one kind of thing fits into another kind of thing. Often, these things are geometric structures. Simple geometric structures include points, line segments, triangular regions, and higher-dimensional versions of these things called, in general, *n-simplices*. So, a point is a 0-simplex, a line segment is a 1-simplex, and a triangular region is a 2-simplex, and so on. "Fitting" one of these things into another thing is called a *map*. Study of complicated geometrical things beginning by "probing" them with how simpler things map into them is a large branch of mathematics called *algebraic topology*. Maps of more complicated things into large but simple things is another kind of mathematical study. The simplest more complicated thing than a triangular region is a circle – it has a "hole." Study of maps of a circle into large flat spaces has also developed into new areas of mathematics. For example, maps of a circle into three-dimensional flat space (the space of "high school solid geometry") is called *knot theory*. Knot theory is much less about "probing" flat space than about how shapes may be placed in space. Here are examples (see Appendix for code):



In the last few decades knot theory has become entwined with modern physics in surprising, complicated ways. Recently, for example, there is a book called "Gauge Fields, Knots and Gravity" by John Baez and Javier P. Muniain.

Maps of a circle into two-dimensional space is maybe not quite so intellectually stimulating as knot theory, but has also been intensively studied by some mathematicians. It is easy enough to draw a continuous curve on a piece of paper which is "simple" in the sense that it never crosses itself, and is "closed" in the sense that its two ends come together. In this way one maps a circle to a simple closed (continuous) curve in the plane. When one makes a drawing or two like this, one sees that there are three very different parts of the plane on which the figure is drawn. There is the inside, the curve itself, and the outside. The curve is the common boundary of the other two regions, the inside is obviously bounded, and the outside could go on forever. It is possible to draw elaborate simple closed continuous curves for which it is not so easy to tell which points of the plane are inside or outside. Nevertheless, the basic intuition was first formulated as a theorem and proof by mathematician Camille Jordan around the turn of the Nineteenth Century.

Theorem 1. *For any simple closed continuous curve* $\mathbf{B}$ *in the plane* $\mathbf{P}$ *there exists a partition* $\mathbf{P} = \mathbf{E} \cup \mathbf{B} \cup \mathbf{I}$ *such that* $\mathbf{E}$ *is an unbounded open set with boundary* $\mathbf{B}$ *and* $\mathbf{I}$ *is a bounded open set with boundary* $\mathbf{B}$*.*

Although it is easy to state this topological assertion, it is not that easy to prove. But, several proofs, deploying a variety of powerful mathematical technologies, have been contrived: nonstandard analysis, [Nar71], [BC94]; algebraic topology, [Hat02] [Bro06]; computer proof, [Wie08]; discrete geometry, [Nef88], [DO11].

One question that came up was what conditions on a map of the circle into the plane, that is defined by formulas such as polynomials, guarantee that the closed curve is simple. J. W. Alexander, an early inventor of new mathematical ideas about knot theory, published a paper with several sufficient conditions [Ale15], one of which guided the *Mathematica 9.0* code to produce Fig. (1) (see Appendix for code).

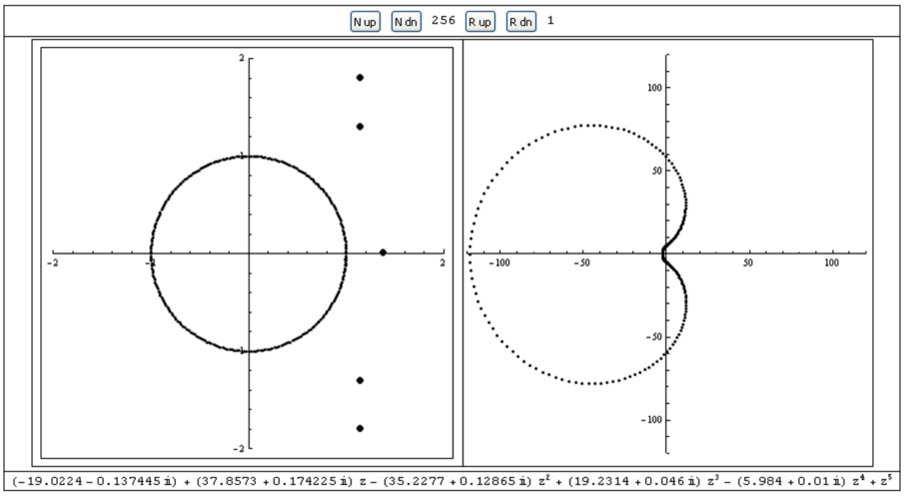


Figure 1: At the very top are buttons for adjusting the number of dots (in this case, 256) for approximating the circle, and for adjusting the radius of the circle (in this case, 1) . The circle is displayed at the left, along with five dots that are the roots of a (complex) polynomial. The figure at the right is the image of the map of the circle into the plane by the polynomial. At the bottom is the fifth degree polynomial (see Appendix for code).

Non-simple maps of the circle into the plane, in other words, where the closed curve may intersect itself, have also been studied. Indeed, the more special case where a non-simple closed curve has no *inflections* – that is to say, points where the curvature is zero – has been studied recently, see Fig. (2) [OOU12].

The purpose of this note is to suggest that it may be interesting to study the simple closed curves that do have inflections.

Two knots in three-dimensional space are considered *equivalent* if one may be continuously deformed – no breaking or cutting – to perfectly overly the other. The question of determining whether two knots are equivalent is difficult, but one approach is to associate with each knot some mathematical structure – usually algebraic structure, such as a polynomial, or a group – in such a way that equivalent knots yield isomorphic structures. Therefore, if the assigned structures of two knots are *not* isomorphic, then the knots are *not* equivalent. Something that stays the same while something else varies is called an *invariant*. So, for example, deforming a knot leaves certain polynomials and certain groups invariant. The paper [OOU12] discloses a new invariant – a non-negative even integer – of closed curves in the plane. In both cases, knots in space and closed curves in the plane, the invariance is

Figure 2: "Closed planar curves without inflections" [OOU12].

relative to continuous deformations.

Define a *shape* in the Euclidean plane to mean simple closed smooth curve with at most double tangents, a finite number of inflection points, and no two equal tangent lines at distinct points of inflection. By "smooth" is meant that the curvature is everywhere defined (involves second derivatives). By "inflection point" is meant a point where the curvature is equal to zero. The simplest but trivial example of a shape is a circle in the plane. Not so trivial is the "bean" shape obtained by smoothly indenting a circle, see the bean shape produced by a polynomial in Fig. (1). The circle and the bean are intuitively "qualitatively distinct." This may be explained in terms of "partial views."

By the Jordan Curve Theorem the plane is partitioned into the exterior of the shape, the shape, and the interior of the shape. A *viewpoint* is a point in the (open) exterior of the shape. If a point source of light is located at a viewpoint, then only a portion of the shape is illuminated. The illuminated portion of the shape is called the *partial view* of the shape from the viewpoint. All partial views of a circle are single connected subsets of the circle. This is not true for the bean shape, nor for the infinite variety of more complicated shapes. In general, a partial view is a finite union of one or more connected subsets of the shape.

I seek to capture algebraically the idea that two shapes are qualitatively similar. If R and S are shapes and Alg(R), Alg(S) are algebraic structures constructed from R, S respectively, the idea is that R is qualitatively similar to S only if Alg(R) and Alg(S) are isomorphic algebraic structures. The basic problem with this idea is that in general, continuous deformation of a shape will almost certainly change its qualitative shape. Witness the polynomial reshaping of a circle shape into a bean shape in Fig. (1). So, whatever may be the algebraic structure Alg(R), it will not be an invariant of continuous deformation. But what sort of deformations should maintain qualitative shape? Certainly, any kind of rigid transformation of the plane – translation, rotation, reflection – would not change qualitative shape.

Computer vision researchers have investigated "characteristic views" of an object (in three dimensional space) for the purpose of recognition. Practical applications include automatic inspection of parts, and motion planning of robot appendages. In any case, alternative constructions of an "aspect graph" data structure may proceed either via orthographic or perspective projections [PD87]. Chapter 7 of my book, "Microlects of Mental Models" [Coo15] includes an algorithm for constructing a diagram in the category of 2-graphs from a shape in the plane. This is based on orthographic projections to planes infinitely removed all around the shape R, so may be denoted $\text{Alg}_{orth}(R)$.

Perspective projections correspond to partial views from nearby viewpoints. Define two viewpoints x and y to be *partial view equivalent* and write $x \sim_{pers} y$ provided there exists a continuous path from x to y such that the number of connected components of partial views along the path is invariant. Paths that cross certain lines in the plane automatically change that number. For example, bitangent and inflection lines contain rays across which that number changes. Not only that, a path that crosses such a ray in one of its two directions will have the same count up to and including viewpoints on the ray, and a different number arbitrarily close to the other side of the ray.This implies there is a directed graph whose dots are partial view equivalence classes of viewpoints, and whose arrows connect two dots if there is exactly one increment in count of some path between the viewpoints in the two equivalence classes. The category generated by this directed graph may be denoted by $\text{Alg}_{pers}(R)$. In summary, there are two algebraic structures $\text{Alg}_{orth}(R)$ and $\text{Alg}_{pers}(R)$ associated with a shape R in the plane, and both seem to be candidates for its "qualitative shape." Hence, I conjecture that the category **Orth** of all $\text{Alg}_{orth}(R)$ is equivalent to the category **Pers** of all $\text{Alg}_{pers}(R)$. To make this conjecture precise, and then to prove it, seem to me great challenges.

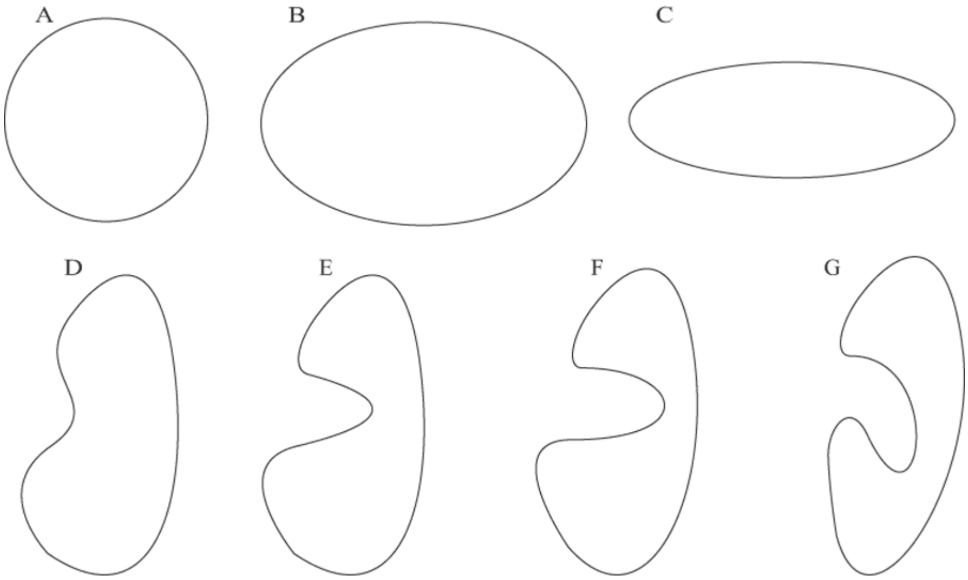


Figure 3: (A) is the circular shape, which is qualitatively simpler than the bean shape. (B) and (C) are quantitatively distinct from the circular shape, but qualitatively similar to it. (D) The simplest non-circular shape, the "bean." (E) and (F) are qualitatively also bean shapes. But (G) is qualitatively different from the bean shape.

Appendix

Mathematica 9.0 code to generate twelve knot images:

```
tbl = Table[
  Graphics3D[{Orange, Specularity[White, 70],
    KnotData[{left[[i]], right[[i]]}, "ImageData"]}, Boxed -> True,
   ViewPoint -> {0, 0.1, 5}],
  {i, 1, 12}]
MatrixForm[Table[tbl[[i + j - 1]], {i, 1, 12, 4}, {j, 1, 4}]]
```

Mathematica 9.0 code to map the unit circle with a polynomial to yield a "bean" shape:

```
StartCircle[pc_, r_] :=
  Table[{r*Cos[i 2 \[Pi]/pc], r*Sin[i 2 \[Pi]/pc]}, {i, 1, pc}];
Nd = 256;
R = 1;
n1 = Nd;
r1 = 1;
rng = 2;
rootcount = 5;
Cmplx[{x_, y_}] := x + I*y;
Pnt[z_Complex] := {Re[z], Im[z]};
Plynml[p_] := Product[(z - Cmplx[p[[i]]]), {i, 1, Length[p]}];
(*pv=RandomReal[{-2,2},{rootcount,2}];*)
(*pv=Table[{1.8,i},{i,-2,2}];*)
pv = {{1.15, -1.8}, {1.15, -1.3}, {1.2, 0}, {1.15, 1.3}, {1.15, 1.8}};
DynamicModule[{},
 btn = Grid[{
    {Button["N up", n1 = 1 + Mod[n1, Nd]],
     Button["N dn", n1 = Mod[n1 - 1, Nd, 1]],
     Dynamic[n1],

     Button["R up", R = Min[4, R + 0.1]],
     Button["R dn", R = Max[0.1, R - 0.1]],
     Dynamic[R]}
    }];
 lpn = LocatorPane[Dynamic[pv]
   ,
   Grid[
    {
     {Show[Graphics[{
         {White, Rectangle[{-rng, -rng}, {rng, rng}]},
         Dynamic[Point[pv]],
```

```
        Dynamic[Point[StartCircle[n1, R]]]
        }]
      , Axes -> True, PlotRange -> {{-rng, rng}, {-rng, rng}},
      ImageSize -> 400]
     }
    }
   , Frame -> All]

  , Appearance -> Table[Style["\[FilledCircle]", Black], {j, Nd}]
  ];
Grid[
 {
  {btn}
  , {lpn}
  , {Dynamic[
     Module[{},
      Show[Graphics[
        Point[Pnt[#] & /@ ((Plynml[pv] /. z -> #) & /@ (Cmplx[#] & /@
               StartCircle[n1, R]))]], Axes -> True,
       PlotRange -> {{-60*rng, 60*rng}, {-60*rng, 60*rng}},
       ImageSize -> 400]]
     ] // N}
  , {Dynamic[MatrixForm[Expand[Chop[Plynml[pv]]] /. Plus -> List]]}
  }
 , Frame -> All]
]
```

Or, for a horizontal arrangement:

```
Grid[{{btn}, {Grid[{{lpn, Dynamic[
      Module[{},
       Show[Graphics[
         Point[Pnt[#] & /@ ((Plynml[pv] /. z -> #) & /@ (Cmplx[#] & /@
                StartCircle[n1, R]))]], Axes -> True,
        PlotRange -> {{-60*rng, 60*rng}, {-60*rng, 60*rng}},
        ImageSize -> 400]]
      ]}}, Frame -> All]}, {Dynamic[Expand[Chop[Plynml[pv]]]]}},
Frame -> All]
```

References

[Ale15] J. W. Alexander. Functions which map the interior of the unit circle upon simple regions. *Annals of Mathematics, Second Series*, 17(1):12–22, September 1915.

[BC94] Nestor Bertoglio and Rolando Chuaqui. An elementary geometric nonstandard proof of the Jordan curve theorem. *Geometriae Dedicata*, 51(1):15–27, May 1994.

[Bro06] Ronald Brown. *Topology and Groupoids.* www.booksurge.com, 2006.

[Coo15] Ellis D. Cooper. *Microlects of Mental Models.* Cognocity Press, 2015.

[DO11] Satyan L. Devadoss and Joseph O'Rourke. *Discrete and Computational Geometry.* Princeton University Press, 2011.

[Hat02] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002. Free download at `http://www.math.cornell.edu/~hatcher/#ATI`.

[Nar71] Louis Narens. A nonstandard proof of the Jordan curve theorem. *Pacific Journal of Mathematics*, 36(1):219–229, 1971.

[Nef88] Lawrence Neffstout. Two discrete forms of the Jordan curve theorem. *American Mathematical Monthly*, 95(4):332–336, April 1988.

[OOU12] Shuntaro Ohno, Tetsuya Ozawa, and Masaaki Umehara. Closed planar curves without inflections. *Proceedings of the American Mathematical Society*, 141(2):651–655, 2012.

[PD87] Harry Plantinga and Charles R. Dyer. Visibility, Occlusion and the Aspect Graph. Computer Sciences Technical Report #736, University of Wisconsin–Madison, December 1987.

[Wie08] Freek Wiedijk. Formal proof - getting started. *Notices of the American Mathematical Society*, 55(11):1408–1414, December 2008. The Jordan Curve Theorem was formally proved twice by independent researchers in 2005.