Diagrammatic Microlects in Computer Science and Mathematics

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ABSTRACT. High school students acquire basic intuition regarding sets by drawing and reasoning about diagrams – Venn diagrams – in which simple closed curves in the plane delimit regions which may or may not overlap. That is enough to learn basic Boolean Algebra, hence Propositional Logic. With additional conventions about representing products of sets, diagrams may represent projections of subsets of products, hence universal and existential quantifiers (and their adjoint properties with respect to "substitution"), whence Quantifier Logic. Dots connected by arrows offer an entirely different species of diagrams and reasoning with intuitive appeal at least as compelling as regions enclosed by curves. Instead of thinking about joining and overlapping of regions, one thinks about compounding connections between locations. As there are rules for calculating and reasoning with Venn regions, there are just as reasonable rules for calculating and reasoning with drawings of dots and arrows. Such rules are what this supplement starts to explore. It is not to suggest a variant *foundation* of mathematics, but just to intimate a firm *ground* upon which *any* foundation might be erected.

Contents

1 Introduction

Mathematicians do not normally discuss their minds. My mental model of mind is of a structure that includes ideas.[1] Some ideas are about how certain things work, and I call those ideas "mental models" or "imagination machines." Since "idea" is a short – and suggestive – word, I will tend to use it interchangeably with the word "mental model." Mental models have histories – some are intrinsic to entire biological species, some evolve gradually in minds, some tend to disappear, but almost always they overlap, interact, or even collide. One salient mental model of "conscious thought" includes a succession of foci of attention, such as the attention you focus on these words.[2] Except for the developing science of mind based on brain measurements with magnetic resonance imaging (MRI) machines (see Chapter 1 of [Coo15]), our only access to ideas in general, and mental models in particular, is through carefully organized physical expressions with words and diagrams and symbolic notations. Organization of expressions includes vocabulary and grammatical rules. This specialized language is called the microlect of the mental model [Coo15]. Microlects are the public faces of private mental models. My microlect of mind includes in its vocabulary the terms

1. mental model
2. imagination machine
3. focus of attention
4. expression
5. microlect

Microlects may be verbal, such as natural language expressions which are one-dimensional sequences of words (and punctuation conventions). They may be diagrammatic, such as plumbing schematics with pipes connecting tanks, pumps, and valves. And they may be mathematical, involving elaborate symbolic expressions with letters, numbers, invented symbols, subscripts, superscripts, and so on. Mathematical expressions are generally, but not necessarily, once-dimensional sequences.[3] The focus of this supplement to [Coo15] is on microlects for a mental model of computer memory. First, there is a verbal microlect with standard vocabulary. In textbooks this microlect is often illustrated with a diagram. Second, the precise diagrammatic microlect of timing machines (see Chapter 2 of [Coo15]) expresses the mental model of computer memory.

Computer memory may be considered an abstract data type. A rigorous foundation for a theory of abstract data types is based on mathematical category theory, especially by Charles Ehresmann, Michael Barr, and Charles Wells

[1] The word "idea" is not intended to suggest any sort of "timeless essence" or "ideal universal."

[2] See Chapters 4 and 5 of [Coo15] about "conscious thought" and "attention."

[3] Stephen Wolfram's programming language, *Mathematica*, is sufficiently nuanced to express any linear *or* non-linear mathematical expression as an equivalent linear expression.

[Ehr65][CL84][WB87][Wel93]. It may be a challenge to gain adequate familiarity with category theory if one needs rigorous understanding. One reason is that the usual presentations of category theory, even those targeted to computer scientists, presume considerable mathematical experience. For example, [BW98] begins with "sets," "functions," "graphs," and "homomorphisms of graphs." If one turns to the "bible" of category theory [Mac71], which is intended for "working mathematicians," the definition of a mathematical category depends at the start on the idea of "composable pairs of arrows" which is explained using a set theory formula, and a diagram of elements of sets to explain "associativity of composition." In other words, preliminary familiarity with mathematical microlects in set theory and category theory *seem* to be prerequisites for a rigorous theory of abstract data types.

So, third, this supplement to [Coo15] suggests an alternative, rigorous grounding for a theory of abstract data types. It takes the form of a a new diagrammatic microlect, with its own vocabulary and grammar for diagrams, with no prerequisites aside from memory, patient focus of attention on detail, and the ability to recognize repeated symbols. But you already know how to do that, you are reading this.

2 Verbal Microlect for Computer Memory

Computer memory is based on a numbered sequence of locations. The numbering starts at 0 and increases by 1 to arrive at the next location. These numbers are addresses of locations. Each location stores a value, which is a number anywhere from 0 to a specified maximum value, such as 255 ($255 = 2^8 - 1$). So far, this verbal microlect includes the terms

1. location
2. address
3. value
4. store

It is interwoven with terms of a mathematical microlect such as

1. sequence
2. 0
3. 1
4. 2
5. 255

This intercalation of terms from distinct microlects signals an interaction of mental models. Computer memory has two other separate locations called

1. address pointer
2. accumulator

The value of the address pointer is an address of a memory location. Calculations – which are not part of the mental model of computer memory *per se* – may yield different values in the address pointer, which therefore refer to different locations in memory. The value of the accumulator may be equal to the value of the memory location to which the address pointer refers, or not. Calculation – also not part of computer memory – may yield a different value stored in the accumulator. There exists a “set” operation that copies the value in the accumulator to the memory location to which the address pointer refers, and there exists a “get” operation that copies the value in the location to which the address pointer refers, to the accumulator. The remaining terms of the verbal microlect are

1. reference
2. copy operation
3. set
4. get

There exists a non-standardized but generally understood diagrammatic microlect for the mental model of computer memory, as in Fig.(1). The advantage over words of not-too-complicated diagrams is that they are like a bird’s-eye view of The Big Picture.

The disadvantage of the diagram in Fig.(1), apart from its *ad hoc* conventions, is that it is not general – particular numbers and arrow positions are necessary in this representation. The timing machine diagrammatic microlect in Chapter 2 of [Coo15] offers a standardized and more abstract diagram. Indeed, Chapter 3 there provides an algorithm, the Micro-Timing Formula,[4] that could simulate computer memory, and much else besides.

[4]For which there is a rudimentary but working Python implementation available at `http://www.cognocity.org`.

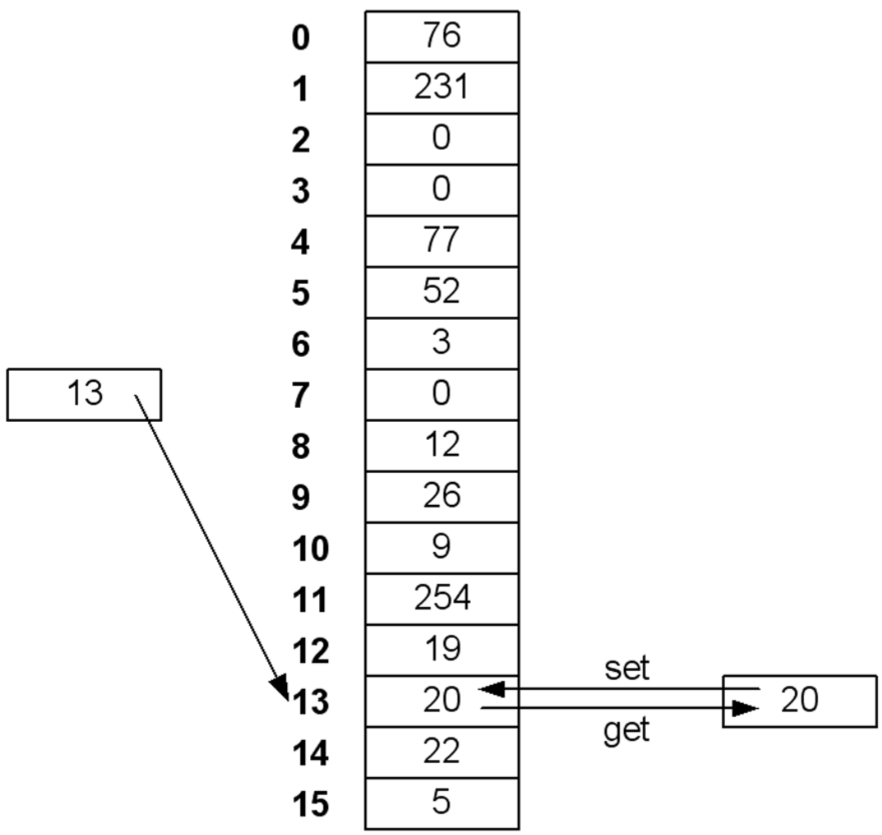


Figure 1: Conventional diagram of computer memory.

3 Timing Machine for Computer Memory

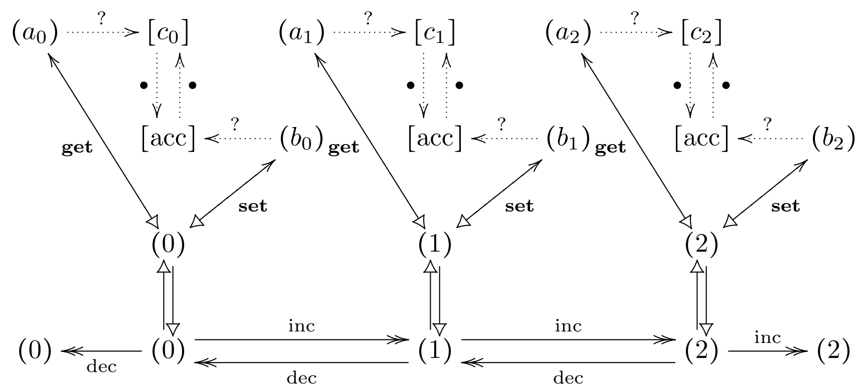


Figure 2: Expression in timing machine diagrammatic microlect of three computer memory cells.

The timing machine in Fig.(2) expresses a mental model of a computer memory with three locations, the variables $[c_0]$, $[c_1]$, and $[c_2]$.[5] More locations could be added by repeating the obvious pattern. The machine has two parts in addition to the three location parts (recall that timing machine parts are isolated but communicate by dotted-shaft signal arrows). The accumulator is a variable [acc], and the address pointer has nine states, (0), (1), (2), (a_0), (a_1), (a_2), and (b_0), (b_1), (b_2). Four signals to the timing machine influence the states of these parts.

An **inc** (respectively, **dec**) signal triggers increments (respectively, decrements) of the address pointer, except at (0) and (2), which bound the address below and above, respectively. By definition of machine part, exactly one of the states (0), (1), (2) is ever active at one time. The timeout arrows $(i) \Leftrightarrow (i+1)$ just signify that each state (i), if active, loops, or "waits," until a signal arrives. If a **set** signal arrives while (i) is active, (i) is triggered to (b_i) which times out quickly back to (i) while emitting a query to [acc], yielding its value to $[c_i]$. Dually, if a **get** signal arrives, (i) is triggered to (a_i) which updates [acc] from $[c_i]$.

If the **inc** signal arrow from (2) to itself at the right is replaced by a countably infinite sequence of rightward **inc** signal arrows, and the diagram above (2) is also duplicated with appropriate changes to correspond with the natural numbers beyond 2, then the resulting infinite timing machine would express the mental model of an infinite computer memory.

[5]For details on timing machine variables, see ([Coo15], p.47).

4 Categorical Diagrammatic Microlect

Calculating or reasoning entirely with diagrams and without elements or equations is performed according to diagram re-write rules. These are two-dimensional analogs of the linear rewrite rules familiar to computer scientists as Backus-Nauer Forms, or to linguists as context-free grammars. The context-free grammar [BKL09] in Fig.(3) generates some sentences of the English language. For example, the calculation

```
S -> NP VP
  -> Det N VP
  -> the N VP
  -> the telescope VP
  -> the telescope V NP
  -> the telescope V Mary
  -> the telescope ate Mary
```

generates the sentence, "The telescope ate Mary." which is not merely false but nonsensical. But it *is* a grammatically correct one-dimensional arrangement of words according to this grammar.

```
S -> NP VP
VP -> V NP | V NP PP
PP -> P NP
V -> "saw" | "ate" | "walked"
NP -> "John" | "Mary" | "Bob" | Det N | Det N PP
Det -> "a" | "an" | "the" | "my"
N -> "man" | "dog" | "cat" | "telescope" | "park"
P -> "in" | "on" | "by" | "with"
```

Figure 3: Example of context-free grammar.

Two-dimensional rewrite rules govern calculation with diagrams formed from symbols and dots and arrows. This activity is most certainly not a reduction of mathematics to a game of playing with symbols. On the contrary, the diagrams of the microlect express ideas which are the pulse of conscious thought. In general, calculating with diagrams is a deep subject. See, for example [AB96] on "diagrammatic calculi" and "diagrammatic logic." In particular, here the idea of calculating with diagrams is inspired by constructions and proofs in mathematical category theory, so this is called a *categorical diagrammatic microlect*.

4.1 Ground for Foundations of Mathematics

Foundations of Mathematics is a rich intellectual discussion amongst mathematicians, philosophers, and historians of mathematics.

> Mathematics in the past century has begun to study its own methods of reasoning and its own structure as the objects of new mathematical methods and disciplines. In the first decade of the twentieth century, in particular, fundamental attitudes towards mathematics were transformed in the discussion of the foundations of mathematics [BL78].

In those early discussions, *foundations of mathematics* divided more or less into three ideologies, Logicism basing mathematics on Logic, Intuitionism basing it on Construction, and Formalism founding it on Symbols [Lam94]. By the turn to the latest century considerably more nuanced and technical developments are at the forefront of the discussion.

Two technical foundations of mathematics are Set Theory and Category Theory, both in the Logicism camp. Almost all mathematicians learn and use Set Theory in most of their work, which was created by Georg Cantor in 1895 [Can55]. F. William Lawvere initiated "The Category of Categories as a Foundation for Mathematics" in 1963-1966 [Law66].[6] He explained there that by "foundation" he means "a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved."[7] A very modern technical foundation for mathematics might be said to attend a new camp, Computationalism, but that is merely my word for the idea promulgated by Vladimir Voevodsky called "Univalent Foundations" of mathematics. It uses a computer application program called *Coq* as a proof assistant in a system based not on logic, not on sets, not on categories, but on Homotopy Type Theory [APW14].

There exists a work in progress by philosopher Elaine Landry called "Categories for the Working Philosopher."[8] Of course, this resonates with the handy reference, "Categories for the Working Mathematician" by Saunders Mac Lane [Mac71]. Recently (April, 2015) Michael Shulman posted a "complete draft" of his contribution, "Homotopy Type Theory: A synthetic approach to higher equalities,"[9] which introduces Univalent Foundations for philosophers. For him, "foundation of mathematics" means "simply that we can encode the rest of mathematics into it somehow," and even more, that we can "translate back and forth to previously existing foundations." Indeed, it has been demonstrated that "any mathematics that can be encoded into set theory can also be encoded" into Higher Order Homotopy Type Theory with Univalent Foundation.

Computer scientists are quite familiar with encoding one programming language in another. A compiler for the **C** language encodes symbolic expres-

[6] More recently, there is [Mak98].

[7] Although based on logic, his proposal was not universally acclaimed, especially among some logicians, for example, George Kreisel. Mathematical philosopher Jean-Pierre Marquis has elaborated on the disagreement.

[8] `https://golem.ph.utexas.edu/category/2013/11/categories_for_the_working_phi.html`.

[9] `https://golem.ph.utexas.edu/category/2015/04/a_synthetic_approach_to_higher.html`.

sions of **C** in the **assembly language** of a particular microprocessor. The microprocessor, in turn, interprets assembly language instructions in the specific **microcode** of the microprocessor. Microcode, in its turn, opens and closes "gates" to manage the flow of electrical currents. At the lowest level in this hierarchy of computer organization [Tan94], are the quantum mechanical band gap properties of semiconductor materials. *But, this level is common ground for all microprocessors, regardless of specific microcodes.* The point of this supplement is that perhaps there also exists a ground beneath all foundations of mathematics. This would be a "physics of mathematical drawings."

For mind to express mental models with microlects the minimum requirements are the ability to invent and copy symbols, and to (re)cognize existing symbols. These physical activities depend on muscular contractions resulting in time-varying force fields (see [Coo11]) applied to physical objects (vocal chords, pencils, keyboards, and so on) and photon detectors – eyes. Operations of largely unconscious mental models connect forces and photons to conscious thoughts.

The most elementary drawing is a **symbol**. A symbol is a type of shape whose tokens have no moving or replaceable parts. For example, the dot over the bottom part of the lowercase letter "i" is not connected to the lower part, but is rigidly at that position, and one does not replace it by another sub-symbol. In this sense, symbols are "atomic," and include letters, digits, punctuation marks, and in mathematics, of course, a host of special symbols such as $\forall$ and $\varnothing$. **Symbolic expressions** are "molecules" compounded from symbols. The symbols are rigidly positioned, but may be replaceable by other symbols, or symbolic expressions. **Diagrams** are compounded from symbolic expressions, but parts are movable and some replaceable. For simplicity of discourse in this supplement, symbols are considered to be symbolic expressions, and symbolic expressions are considered to be diagrams.

Categorical diagrammatics proceeds in units of more or less prolonged focus of attention called *discourses* that combine text and diagrams. The text is to the diagrams as comments in a computer program are to the actual code – helpful epiphenomena. Successive discourses may be distinguished by page-wide horizontal lines. A symbol is *new (for a discourse)* provided it does not appear in the discourse, even if it appears in other discourses. The following symbols are *reserved symbols* and may not be new symbols for a discourse.

$$\bullet$$
$$\not\bullet$$
$$1$$
$$\rightharpoonup$$
$$:\rightharpoonup$$
$$\longrightarrow$$
$$.$$
$$\blacktriangleleft$$
$$\blacktriangleright$$

The $\bullet$ symbol expresses a mental model of anonymous, *primordial wholeness*, a single, whole, un-named item. The $\not\bullet$ symbol expresses *primordial emptiness*. The arrow $\rightharpoonup$ expresses the possibility of introducing a *new diagram* into the discourse, which appears at its head end, based on a diagram already appearing in the discourse and shown at the tail end. The arrow $:\rightharpoonup$ expresses the option to copy an existing diagram *but* with copies of the diagram at its head end replacing occurrences of symbols which are the same as its tail end. The tail end is an *abbreviation* for the head end.[10] These half-headed arrows are *ground rules* for advancing discourse by diagram introduction or replacement. Ground rules are to mathematics as *Labanotation* is to dance.

The idea "from this, to that," of *primordial directionality* is expressed by the full-headed arrow symbol

$$\longrightarrow$$

which is the distinctive, characteristic symbol of categorical diagrammatics. Throughout most of the world roads have painted arrows indicating legal direction of traffic flow. Flow charts in computer science use arrows to represent direction of control or data flow. In Husserlian phenomenology, "intentionality" is the directionality of "acts of consciousness." And category theory, of course, is the quintessential mathematical discipline using diagrams with arrows to represent the direction of morphisms, or functions, or transitions, or transformations.

The reserved symbols $\blacktriangleleft$, $\blacktriangleright$ are explained later.

Ground rules for reserved symbols to create and grow discourses define their meanings. As in type theory, there are no axioms or rules for logical deduction. "It is important to understand that these *rules* are not the same sort of thing as the *axioms* of a theory like [axiomatic set theory] or [elementary theory of the

[10] Such rules correspond to "judgemental" or "definitional equalities" in type theory [TUFP13].

category of sets]. Axioms are statements *inside* ambient superstructure of (usually first-order) logic, whereas the rules of type theory exist at the same level as the deductive system of logic itself." As a foundation of mathematics, "type theory is "closer to the bottom" than set theory: rather than building on the same "sub-foundations" (first-order logic), we "re-excavate" the sub-foundations and incorporate them into the foundational theory itelf." [11] I am arguing that foundations and sub-foundations, excavations and re-excavations, all rest upon or in a common ground. I would not expect you to accept such a claim without sufficient evidence, and the best evidence would probably be along the lines of showing how all of Set Theory, Category Theory, and Univalent Foundations can be encoded in a categorical diagrammatic microlect. This may be done in the future, but for now the best I may do is indicate by some examples what this might look like some day.

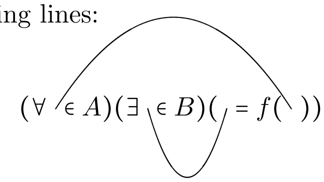
In general, diagrams are drawings of these "jangly" contraptions of dots connected by arrows. However, it must be understood that "dot" really stands for a symbolic expression, which is the tail or head of one or more arrows. An arrow is never without a label, which is also a symbolic expression. Diagrams and Dots and arrows are public physical structures that express private *mental models of structures*. In particular, arrows express the directed relationships such as (but not only) transformations between structures expressed by dots. The special diagram $\bullet \xrightarrow{x} A$ expresses the idea of a *structural element* x of structure A. A structure A may have no, one, or more than one structural element, and it may be the head of one or more arrows whose tails are not $\bullet$. Also, a structural element x may itself have structural elements, as in $\bullet \xrightarrow{K} x$, and so on.

The particular symbols that occupy dot positions, or the label symbols at positions beside arrows, have no purpose other than to identify positions where symbols may be "uniformly substituted." "Physically substituting" for a symbol at a position suggests erasure of the symbol at the position, followed by insertion of a copy of another symbol at the blank position. This is a subtractive operation followed by an additive operation. An alternative to "substitution" is "replacement," which is to copy the entire diagram except for a symbol at a position, and to copy another symbol at the designated position. This is a purely additive operation. Either "substitution" or "replacement" leads to the same new diagram in discourse. The word "uniform" means that multiple occurrences of the same symbol at different positions in a diagram may be replaced by copies of the same other symbol.

In conventional mathematical microlects, a "variable" is a symbol which may be given a "value" by replacement. Every variable occurs within the "scope" of a "binder" such as the operators $\forall\cdots$ or $\exists\cdots$ or $\{\cdots|\cdots\}$ or $\lambda\cdots:\cdots$. There are technical rules for replacing symbols for variables by other symbols of variables,

[11] Michael Shulman, 2015, "Homotopy Type Theory: A synthetic approach to higher equalities" `https://golem.ph.utexas.edu/category/2015/04/a_synthetic_approach_to_higher.html`.

or by "constants." These symbols and their rules, including rules for deriving new symbolic expressions from previously assumed or derived expressions, form an extremely successful notational ground for conventional mathematics. There is also a kind of "cognitive overload" in this notational system, since all occurrences of a variable associated with a particular binding operator may be replaced by any other symbol, subject to restrictions about avoiding binding collisions. For example, in $(\forall x \in A)(\exists y \in B)(y = f(x))$ the variable x is bound by $(\forall x \in A)$, and both occurrences of x may be replaced by occurrences of z to derive the entirely equivalent expression $(\forall z \in A)(\exists y \in B)(y = f(z))$, but not by occurrences of y to derive $(\forall y \in A)(\exists y \in B)(y = f(y))$, since although y is in the scope of $(\forall x \in A)$, it is also in the inner scope of $(\exists y \in B)$. All such complexity arises basically because of the general convenience of calculating with first-order formulas written as linear expressions. That is to say, if planar expressions were the norm then *ligatures* could replace variables entirely,[12] without loss of logical significance, and gain in simplicity of expression, but at the expense of drawing diagrams instead of writing lines:



Even the so-called reserved symbols, such as the binding operators, are themselves bound by the wider discourse of mathematical conventions. In other words, in both computer science and mathematics, *every symbol is bound in a sufficiently wide context.*[13]

The verbal microlect of the above mental model of categorical diagrammatic discourse includes at least the following vocabulary items:

1. discourse
2. new symbol
3. wholeness
4. emptiness
5. identical structure
6. tail
7. shaft
8. head

[12] Ligatures in Peirce's existential graphs [Dau11] are more difficult to interpret.

[13] In physics, on the other hand, variables in equations may carry substantially more significance than mere options for substitution or replacement. Such variables are routinely associated with "units of measurement," such as Volts or Ohms or Amperes. But it is not generally insisted that "units of measurement" are themselves associated with "processes of measurement," which are physical movements of experimenters in laboratories involving physical objects according to precise orchestrations. Whether a post-graduate student in the laboratory walks to an oven starting with the left foot or with the right foot is generally irrelevant, so the physical motions of experimenters are equivalent "up-to-homotopy." Correct operation of a pipette is more demanding, so its motion homotopy class is relatively smaller.

9. arrow
10. directionality
11. abbreviation
12. occurrence
13. substitute
14. replace
15. copy

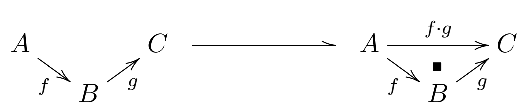
4.2 Identity Rule

Beyond the rules for merely re-arranging diagrams at the ground level, there are rules for creating essentially new diagrams. A three part diagram with $\longrightarrow$ in the middle is a *ground rule* for advancing discourse. Rules may introduce new rules, and a discourse without at least one initial diagram and one rule cannot develop. So, a discourse, in general, is a "deduction," as understood, say, in type theory ([APW14], p.602).

$$A \longrightarrow A \xrightarrow{1_A} A \qquad (1)$$

Every symbol has an "identity arrow." The rule (1) says that for any symbol a diagram may be introduced to discourse with an arrow from a copy of the symbol to another copy of the symbol. That arrow shall be labeled with the reserved symbol 1 subscripted by a copy of the symbol. The arrow so labeled is called the *identity arrow* of the symbol. (Note that in this discourse there occur four copies of the symbol, A, and one occurrence of each of three reserved symbols.) The identity arrow expresses the idea of the operation or function or transformation that preserves the identity of its tail end symbol. It also provides an arrow for every dot with neither gain nor loss of information in the discourse, but with a gain in expressiveness, as will be appreciated.

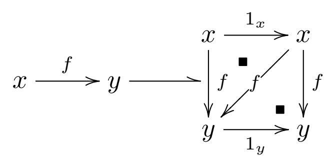
4.3 Composition Rule



(2)

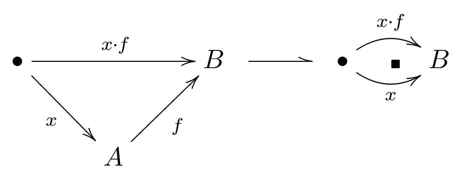
Any diagram with two arrows connected so that the head symbol of an arrow is the same as the tail symbol of an arrow may be completed to a triangular diagram containing the reserved symbol ▪ meaning *same* plus the third side labeled by the label of the first arrow followed immediately by the reserved symbol · , and followed by the label of the other arrow. If several arrows $f_1, f_2, \ldots, f_n$ are connected chain-like with head of one the *same* as the tail of the next, then a new arrow $f_1 \cdot f_2 \cdot \ldots \cdot f_n$ may be added to the discourse. Such chains of arrows are called *compositions*. Compositions express the idea of doing or performing or executing operations or transitions or transformations one after another.

Identity arrows conform to a ground rule with respect to composition, namely



(3)

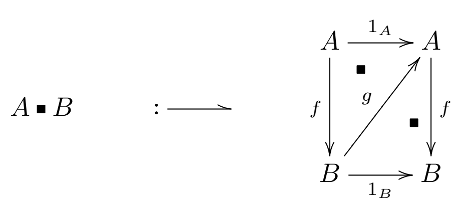
If there is a diagram $\bullet \xrightarrow{x} A \xrightarrow{f} B$, then by the Composition Rule there is a diagram $\bullet \xrightarrow{x \cdot f} B$, which expresses the idea of x transformed by f with resulting structural element $x \cdot f$ of B. Of course, the transform $x \cdot f$ may not be identical to x. But, if $A \xrightarrow{f} B$ and it so happens that every structural element $\bullet \xrightarrow{x} A$ is also a structural element of B, so that



(4)

then f is called an *inclusion* of structure A in B. For any other inclusion g of A in B, x and $x \cdot g$ are also the same, hence by the transitivity of sameness, $x \cdot f$ and $x \cdot g$ are the same. For that reason, there is exactly one inclusion of A in B insofar as structural elements are concerned, hence a single symbol $A \xrightarrow{\mu_B^A} B$ may stand for *that* inclusion.

4.4 Same Structure



(5)

The idea of *sameness* for structures is that two symbols represent the same structure if there are transformations one to the other in both directions whose compositions – which are loops – change nothing. The identity arrows represent "no change." The symbol ▪ is pronounced "same (structure)."[14] The symbol ▪ is used both for same structure and same path.

4.5 Congruence Rule

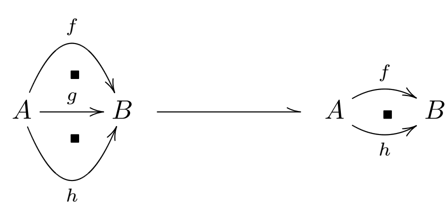
If all corresponding parts of two diagrams are the same, then the diagrams are the same. This is analogous to the definition of congruence of geometric figures that high school students know.

4.6 Interchange Rule

(6)

In rule (6) the right side diagram may only be re-written as the left if the head of a and the head of b are the *same*.

4.7 Deletion Rule



(7)

The reserved symbol ▪ expresses "sameness" with regard to pathways of transition between symbols. In this rule, assuming path by arrow f is the same as the path by arrow g, and that the path by arrow g and the path by arrow h are also the same, then the path by arrow f is the same as the path by h. Otherwise said, this ground rule is the transitivity of path sameness.

[14] This definition of *same* with a diagram is motivated by a pattern that goes back to the definition of "same cardinality" of two sets. That pattern is carried forward to diverse categories under the name "isomorphism of objects."

The category theorist would consider a diagram filled with a symbol such as ▪ to be a "commutative diagram." This concept depends crucially on a notion of "equality" between elements of a "hom set" in the category where the paths are composed of morphisms. But the notion of equality of set elements, not to mention of category, according to this supplement, are not necessary for the idea of "same path," which is the same idea in all categories. In other words, ▪ for structure and ▪ for directed connection between structures, are what the computer scientist calls "polymorphic." A mathematician might just say, they are both equivalence relations – but that is a concept of set theory, and neither set theory nor category theory are prerequisites for grounding foundations of mathematics.

4.8 Outside-Inside Rule

(8)

On some occasions in discourse, it may develop that in a diagram it is assumed or calculated that two outside paths of arrows are the same, where both have the same tail and the same head. In such a case the ▪ symbol may not be placed inside the region bounded by those outside paths, since there may be inside paths. In that case, a ▪ symbol in the immediate vicinity of one of the outside paths indicates that it is the same as the other one.

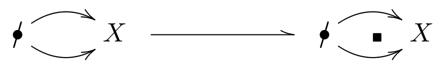
4.9 Terminal Rules

(9)

$$X \longrightarrow \bullet \qquad \longrightarrow \qquad X \xrightarrow{\tau_X} \bullet \tag{10}$$

The consideration of a symbol, whatever be the mental model which it expresses, as a single, undifferentiated whole is codified in an arrow from the symbol to the primordial wholeness, $\bullet$. (9) rules that there exists only one way to reduce something to a single undifferentiated wholeness, and (10) gives that way a name. Since there is no possibility of further reduction, these are called "terminal rules," and $\bullet$ is the terminal structure.

4.10 Initial Dot Rules

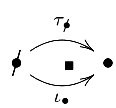


(11)

$$\not\bullet \longrightarrow X \quad \xrightarrow{\qquad\qquad} \quad \not\bullet \xrightarrow{\iota_X} X \tag{12}$$

Likewise, transition from pure emptiness $\not\bullet$ to an arbitrary structure in one swell foop can be imagined, but in only one way, which is expressed by rule (11), and rule (12) names that transition.

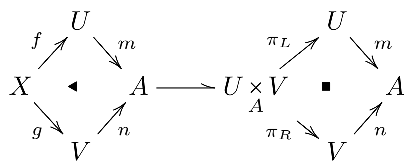
Primordial emptiness and wholeness are connected by two arrows, and it should not matter which one is the path. This sameness is conveyed by the axiomatic diagram



(13)

4.11 Pullback Rules

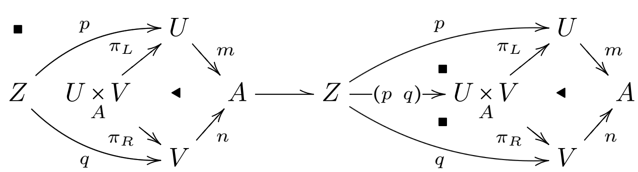
4.11.1 First Rule



(14)

A recurrent theme in category theory is the concept of a "limit diagram," and the simplest example is the "pullback." The left side of (14) uses the reserved symbol $\blacktriangleleft$ to signify that the dots and arrows form a *pullback diagram.* The first rule about a pullback diagram is that there is a standard symbol $U \underset{A}{\times} V$ for the dot and two arrows π_L and π_R from it, and that the two paths are the same. The symbols L and R stand for *left* and *right*, and are used even if $U \blacksquare V$.

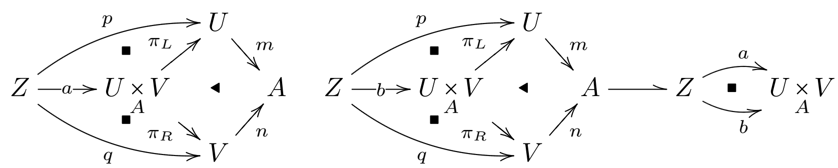
4.11.2 Second Rule



(15)

The second rule is the heart of the limit idea in this case: if any two arrows p and q from any Z partake in two same outer paths as indicated by the outside $\blacksquare$ on the left, then there exists a "fill-in" arrow from Z to X shared by two pairs of same paths at the right. A mental model of this limit diagram imagines $U \underset{A}{\times} V$, with π_L and π_R, to be the "tightest" way to complete a square whose other two sides are m and n. "Tightest" is formalized in terms of saying that any other completion "factors through" a new arrow from Z to $U \underset{A}{\times} V$.

4.11.3 Final Rule

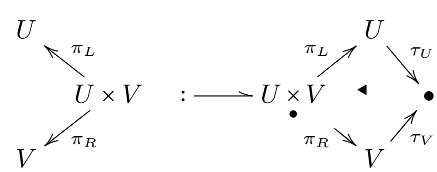


(16)

More, any two ways that any Z, p, q factors through $U \underset{A}{\times} V$ to complete the outer square are the same. This sameness is the final pullback rule.

4.11.4 Product Abbreviation

Certain special cases of the pullback rules are so common that they have their own names and specialized notation. For example, if A is primordial wholeness $\bullet$, then there is no need to include the terminal arrows τ_U and τ_V in the diagram, nor is there a need to include notation for "over $\bullet$." In this case, the pullback of U and V over $\bullet$ is just called the *product* of U and V, and the abbreviation is summarized in (17). The idea of a product is based on the idea of projecting a point in a Cartesian coordinate system upon its two independent coordinates. More generally, a pullback has projections onto coordinates that are not quite independent – they are constrained by the arrows to A.



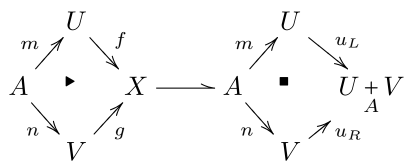
(17)

4.12 Pushout Rules

The entire discourse about Pullback Rules has a thorough analog with all arrows reversed in direction, and appropriate changes in terminology. For example, "limit" changes to "colimit." Also, primordial wholeness is replaced by primordial emptiness. This "dual" discourse is about the *pushout* diagrams, their rules, notations, and the special case called "coproduct." The mental model of the pushout of two structures U and V over A is that of "gluing" U to V "along"

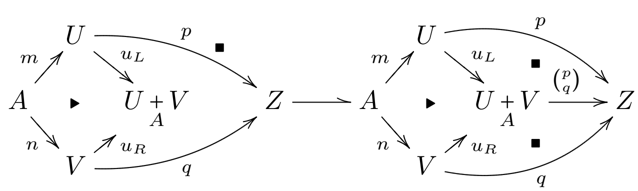
A. The mental model of coproduct of two structures, in particular, is the idea of adding them together to construct a larger structure into which they both fit without touching or interacting or interfering one another.

4.12.1 First Rule



(18)

4.12.2 Second Rule

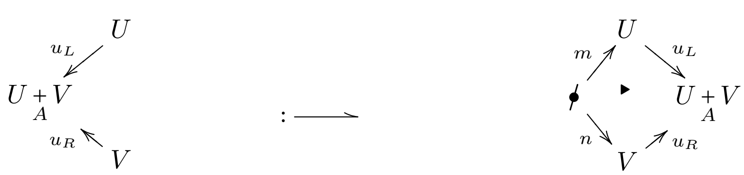


(19)

4.12.3 Final Rule

More, and exactly dual to the Final Rule for pullbacks, any two ways a and b, that Z, p, q factors through $U \underset{A}{+} V$ to complete the outer square, are the same. This is the final pushout rule.

4.12.4 Coproduct Abbreviation



(20)

As product is a special case of pullback, there is an abbreviation called *coproduct* for pushout over the initial dot.

5 Structure Diagrams

5.1 Directed-Graph

The idea of a diagram is that dots are connected by arrows, and that each arrow has a tail dot and a head dot. This mental model *itself* is expressed by a diagram as in (21).

$$A \underset{h}{\overset{t}{\rightrightarrows}} D \tag{21}$$

A verbal microlect for this diagram says that A and D are structures related by two transformations, t and h. One imagines A stands for all the arrows of a diagram, D stands for all the dots, and that t is the transformation that associates with an arrow its tail dot, and h yields the head dot. The official name for any structure with this pattern is **directed graph**. Other names for the same structure are "oriented graph" and "network." Computer science and mathematics are thoroughly shot through with directed graphs.

The structure (21) expresses a mental model of a diagram in which all dots are distinct and all arrows have distinct labels. The more general mental model in which more than one dot may carry the same symbol, or more than one arrow may have the same label, is expressed by the structure

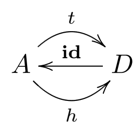
$$\begin{array}{ccc} A & \xrightarrow{t} & D \\ {\scriptstyle F}\downarrow & \blacksquare & \downarrow{\scriptstyle G} \\ L & \xrightarrow[\sigma]{} & S \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{h} & D \\ {\scriptstyle F}\downarrow & \blacksquare & \downarrow{\scriptstyle G} \\ L & \xrightarrow[\tau]{} & S \end{array} \tag{22}$$

in which the upper directed graph defines the "shape" and the lower directed graph defines the "labels" L and the symbols S. The declaration in (22) that $t \cdot G$ is the same as $F \cdot \sigma$ expresses the obvious aspect of the mental model that if two arrows in the shape happen to be labeled the same way by F, then they must also have the same tail, and likewise for heads according to $h \cdot G$. This preservation of tail and head aspects generalizes to the idea of diagrams that preserve other kinds of structure. In category theory one calls (22) a *directed graph morphism.*

5.2 Looped Graph

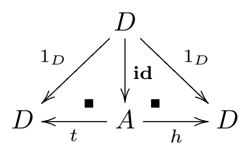
The Identity Rule for arbitrary dots and arrows may be reflected "inside" the directed graph structure by adding an arrow and a diagram. That is to say, in

(23) the additional arrow **id** expresses the mental model of assigning one arrow to each dot.



(23)

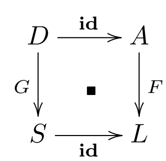
The further constraint, that the tail and head of the assigned arrow shall be just that dot, is expressed by (24).



(24)

Note that this expression of a mental model of directed graph (or "diagram scheme" as in [Mac71]) "with loops" does not require mention of "sets," or "functions," or "elements," or terms of any other conventional mathematical microlect. Of course, (24) uses identity-arrows of the categorical diagrammatic microlect to qualify the **id**-arrow.

The morphism structure for looped graph adds to the directed graph morphism structure (22) a third diagram,

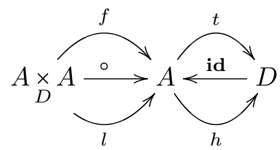


(25)

which expresses the idea of loop preservation.

5.3 Composition Graph

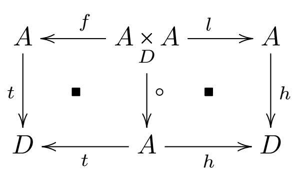
With that success at expressing a mental model of a directed graph with loops, where "external" identity arrows 1 are reflected by **id**, more can be reflected. Indeed, the composition "·" of arrows may be captured by new structure added to the looped graph (23) and (26):



(26)

In discourse with a directed graph determined by arrows A and dots D, the Pullback Rules sanction a new structure with the product of A by itself over D. The two projections are called f for "first" and l for "last," which refer to two "composable" arrows of the graph. For such a derived structure there is postulated a new arrow $A \underset{D}{\times} A \xrightarrow{\circ} A$ expressing the internal "law of composition" for chains of two arrows.

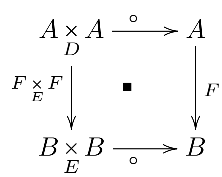
The constraint, that the tail (respectively, head) of a composition is exactly the tail (respectively, head) of the first (respectively, last) arrow, is expressed by diagram (27).



(27)

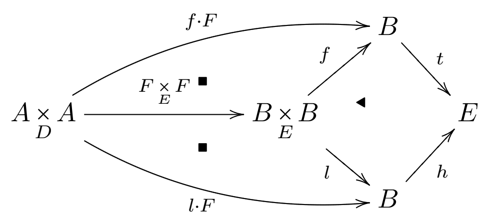
The structure (23)-(24) and(26)-(27) here called a *composition graph*, is closely related to, but not exactly the same, as the concept of *graphe multiplicatif* invented by Charles Ehresmann fifty years ago (See [CL84]).

As before, the idea of morphism is a diagram



(28)

that expresses preservation of the "law of composition," where $F \underset{E}{\times} F$ is determined by the pullback diagram

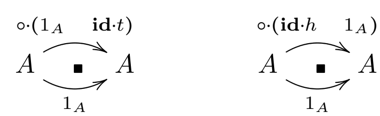


(29)

(30)

5.4 Category

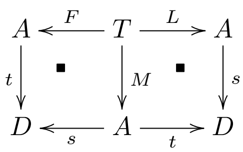
The mental model of a category is the structure of composition graph, augmented by conditions on **id** and $\circ$. Internal "identity rules" constrain the composition in one intuitive way, and an "associativity rule" constrains it another way. The identity rules are



(31)

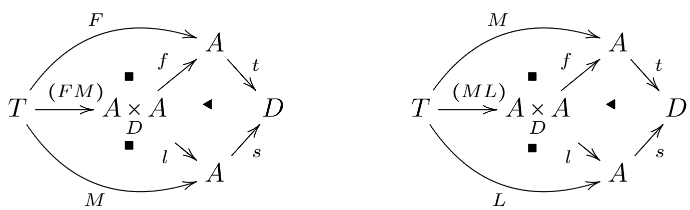
The "associativity rule" is a tad more interesting to express. There are several approaches. One way to express it appeals to a conventional set theory microlect for expressing the set of triples of composable arrows, compose the composition of the first two with the third, compose the first with the composition of the second two, and assert equality of the two results. For example, this is the implied approach in ([Mac71], p.7) in terms of "configurations" consisting of three consecutive arrows, and expressed pictorially in terms of the labels on those three arrows. Second, more in keeping with the "philosophy" of the categorical diagrammatic microlect served here, one could extend the idea of pullbacks – used to construct $A \underset{D}{\times} A$ – to limits of diagrams that yield a triple product of A over D, as in $A \underset{D}{\times} A \underset{D}{\times} A$, which would have three projections F for "first," M for "middle," and L for "last" of the three arrows. Third, one could stay with pullbacks, but iterate to construct $(A \underset{D}{\times} A) \underset{D}{\times} A$ and $A \underset{D}{\times} (A \underset{D}{\times} A)$, and prove these are the same. Fourth, the approach adopted for now is to introduce a new structural element T and three arrows F, M, L to express the "configuration" required for the associativity rule.

For this discourse assume there is a diagram for T as in (32).



(32)

Using pullback rules, construct the diagrams for projecting to the first two (respectively, last two) arrows in a "configuration."



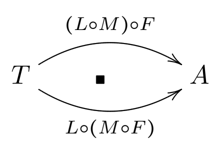
(33)

and then introduce abbreviations

$$M \circ F :\longrightarrow\ (FM)\cdot\circ \tag{34}$$

$$L \circ M :\longrightarrow\ (ML)\cdot\circ \tag{35}$$

The "Associativity Rule" for defining a category is, then, the diagram



(36)

expressing that the two compound compositions are the same, where the compositions with $\circ F$ and $L\circ$ are abbreviations for similar derivations as in (33). A morphism of categories is called a *functor*, and is exactly a morphism of the "underlying" composition graph – because only conditions, not data, distinguish categories from composition graphs.

6 Computer Memory

The mental model of computer memory in this supplement has two parts. On one hand, there is a memory array of (possibly countably many) locations with an address pointer which may be decremented but bounded by 0 below, and incremented (possibly without limit). On the other, there is access interaction – between an addressed memory location and the accumulator – mediated by "set" and "get" operations.

6.1 Address Pointer

As in the use of "sketches" for formal description of data types [WB87], the memory locations form a structure with data diagrams

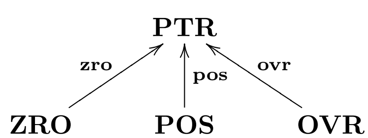
$$\mathbf{PTR} : \longrightarrow \mathbf{ZRO} + \mathbf{POS} + \mathbf{OVR} \tag{37}$$

$$\bullet \blacksquare \mathbf{ZRO} \tag{38}$$

$$\mathbf{ZRO} + \mathbf{POS} \xrightarrow{\mathbf{inc}} \mathbf{POS} + \mathbf{OVR} \tag{39}$$

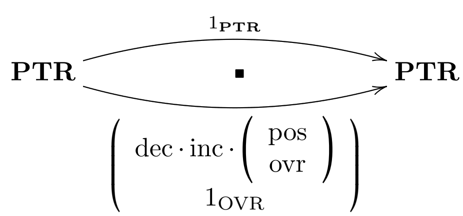
$$\mathbf{POS} \xrightarrow{\mathbf{dec}} \mathbf{ZRO} + \mathbf{POS} \tag{40}$$

a coproduct diagram,



(41)

and a condition diagram



(42)

This takes advantage of the row and column notations for arrows induced by pullbacks and pushouts: the increment **inc** operation "undoes" the decrement **dec** operation (but not the reverse, since **inc** leads to **OVR** for a finite memory, and there is no coming back from there).

6.2 Memory Access

The "set" and "get" operations of the computer memory mental model are expressed by data diagrams

$$\mathbf{ACC} \times \mathbf{PTR} \xrightarrow{\mathbf{set}} \mathbf{MEM} \tag{43}$$

$$\mathbf{MEM} \times \mathbf{PTR} \xrightarrow{\mathbf{get}} \mathbf{ACC} \tag{44}$$

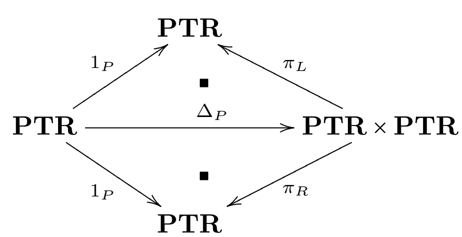
$$\tag{45}$$

constrained by the condition diagrams

$$\begin{array}{ccc} \mathbf{ACC} \times \mathbf{PTR} & \xrightarrow{\quad p_A \quad} & \mathbf{ACC} \\ {\scriptstyle (\mathbf{id_A} \quad \Delta_P)} \downarrow & \blacksquare & \uparrow {\scriptstyle \mathbf{get}} \\ \mathbf{ACC} \times \mathbf{PTR} \times \mathbf{PTR} & \xrightarrow[(\mathbf{set} \quad \mathbf{id}_P)]{} & \mathbf{MEM} \times \mathbf{PTR} \end{array} \tag{46}$$

$$\begin{array}{ccc} \mathbf{MEM} \times \mathbf{PTR} & \xrightarrow{\quad p_M \quad} & \mathbf{ACC} \\ {\scriptstyle (\mathbf{id_A} \quad \Delta_P)} \downarrow & \blacksquare & \uparrow {\scriptstyle \mathbf{set}} \\ \mathbf{MEM} \times \mathbf{PTR} \times \mathbf{PTR} & \xrightarrow[(\mathbf{get} \quad \mathbf{id}_P)]{} & \mathbf{ACC} \times \mathbf{PTR} \end{array} \tag{47}$$

where Δ_P (for "diagonal") is induced by the product diagram



7 Concluding Remarks

A new microlect is introduced to possibly serve as ground upon which alternative foundations for mathematics may be assembled. It is a vocabulary and grammar of two-dimensional rewrite rules for diagrams that express mathematical mental models. These diagrams participate in discourses, and to demonstrate expressiveness of the new microlect, an example from computer science is discussed. This is the concept of computer memory equipped with an address pointer, an accumulator register, and operations "set" and "get" to copy data between memory locations and the accumulator.

A considerable part of mathematics might be grounded without any appeal whatsoever to either naive [Hal60] or axiomatic set theory [Sup60]. For example, the ubiquitous concepts of directed graph and mathematical category are rigorously defined expressions of the new categorical diagrammatic microlect for well-known mental models. It may be possible to express all concepts of categorical logic [Law69] or homotopy type theory [APW14] in terms of the new microlect, but that remains to be seen.

There exists a research community of mathematicians and philosophers of mathematics propelling itself from "the shores of extensional mathematics"[15] that surround "Cantor's Paradise." These metaphors refer to the widely accepted idea that all of mathematics may be "encoded" in the terms of set theory, in other words, that Axiomatic Set Theory as a particular example of mathematical logic is a Foundation for Mathematics. However, this presumption is increasingly under examination, especially with the advent of Category Theory, which is not only difficult to "encode" in set theory, but also opens avenues to broader vistas of Foundations of Mathematics.[16] The most vigorous current work that tends to encompass and extend prior sojourns into uncharted waters involves re-inventing the very idea of equality, which for sets merely says "two sets are *equal* if and only if they have the same elements." The new developments arise in contexts where complicated structures do not have elements to begin with, and equality is itself a complicated structure.[17] This supplement may *appear* to be a rowboat on those same waters, but that is not quite what is intended.

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